Local and Global Stability of Symmetric Heterogeneously-Delayed Control Systems

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Outline

- Stability and Delays
- Kelly Controls
  - Classic Kelly Controls
  - Max-min Kelly Control (MKC)

background

- Heterogeneous Local Stability
- Homogeneous Global Stability
- Conclusions

our work

AQM Congestion Control

- Future high-speed networks are likely to require new types of congestion control
  - Current efforts include XCP, BIC-TCP, FAST TCP, HSTCP, Scalable TCP, etc.
- Besides improving classical E2E approaches, another direction is to involve Active Queue Management (AQM)
  - In AQM, routers compute explicit feedback
  - No per flow management is usually allowed
  - Feedback is computed based on <u>aggregate</u> arrival rates of all flows

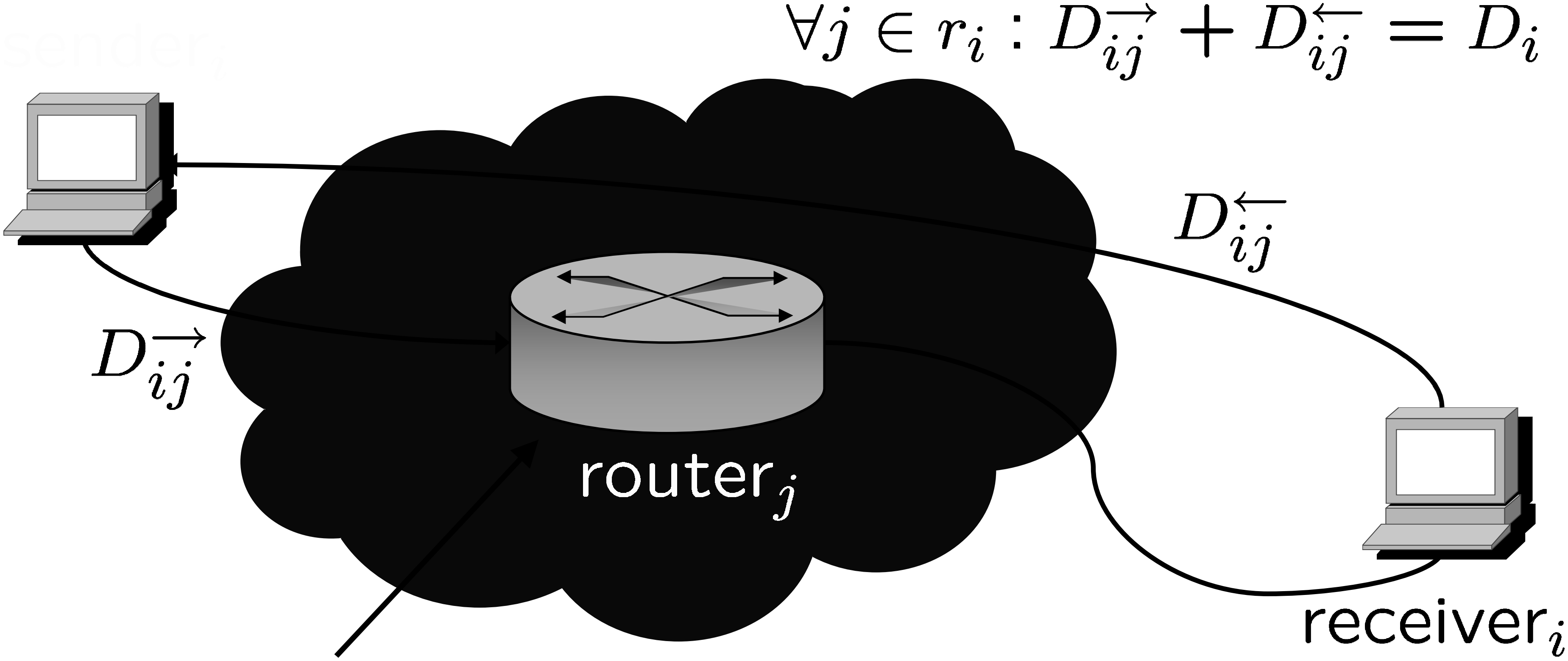
Stability and Delays

- In AQM congestion control, asymptotic stability is one of the most fundamental requirements
- Stability is often compromised by feedback delay
- Delayed stability proofs are generally complicated, especially under heterogeneous delay:
  - Each flow has a different RTT equal to D_i time units
  - Metric D_i can be fixed for each flow or changing over time (i.e., random)

Heterogeneous Directional Delays

- Not only are real Internet delays heterogeneous, they are also directional
  - Delays to/from each router are non-negligible

$$\forall j \in r_i : D_{ij}^{\rightarrow} + D_{ij}^{\leftarrow} = \overset{\text{RTT}}{\overset{\downarrow}{D_i}}$$



For each router j in user's path r_i

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Classic Kelly Control

- Our analysis examines optimization-based framework introduced by Kelly *et al*. in 1998
  - Performance of the system is optimized when the utilities of end-users are locally maximized
- Continuous control has been proven to be globally asymptotically stable in the absence of delay (Kelly 1998)
  - Further analysis under delay has become an active research field (Massoulie 2002, Kunniyur 2000, 2001, 2003, Vinnicombe 2000, 2002, etc.)

Classic Kelly Control 2

- Stability of Kelly control in the discrete case is studied by Johari in 2001
  - Since all real networks are discrete, we also take this approach
- Under heterogeneous feedback delays, Johari *et al.* discretize Kelly control as follows:

$$\begin{aligned} x_i(n) &= x_i(n-1) \longleftarrow \text{preceding sending rate} \\ \uparrow \quad &+ \underbrace{\kappa_i(\omega_i}_{} - x_i(n-D_i) \sum_{j\in r_i} \mu_j(n-D_{ij}^{\leftarrow})), \end{aligned}$$

next sending rate

positive constants

rate RTT time units earlier

over all routers in the path

packet loss of router j

Classic Kelly Control 3

- Assuming $D_i = D$, the discrete Kelly control is locally asymptotically stable if (Johari 2001):

$$\begin{aligned} x_i(n) &= x_i(n-1) \\ &+ \kappa_i(\omega_i - x_i(n-D_i)\sum_{j\in r_i}\mu_j(n-D_{\overleftarrow{ij}})), \end{aligned}$$

where x_u^* is the steady-state rate of user u

- Under heterogeneous delays, continuous Kelly control is locally stable if (Vinnicombe 2000):

$$\kappa_i \sum_{j\in r_i}((p_j + p'_j \sum_{u\in s_j} x_u)|_{x_u^*}) < \frac{\pi}{2D_i}$$

cannot
support
arbitrarily
large delay!

Max-min Kelly Control (MKC)

- End-user equation (SIGCOMM 2004):

$$x_i(n) = (1 - \beta\eta_i(n))x_i(n - D_i) + \alpha$$

constant

packet loss

rate RTT time units earlier

constant

- Utilize max-min fairness, where the feedback is the packet loss of the most-congested resource along the path:

$$\eta_i(n) = \max_{j \in r_i} p_j(n - D_{ij}^{\leftarrow}),$$

set of routers in the path

where:

$$p_j(n) = p_j(\sum_{u \in s_j} x_u(n - D_{uj}^{\rightarrow}))$$

aggregate rate

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Delay-Independent Stability

- Theorem. Assume an N-dimensional undelayed nonlinear system $\mathcal{N}$:

$$x_i(n) = f_i(x_1(n-1), x_2(n-1), \cdots, x_N(n-1)),$$

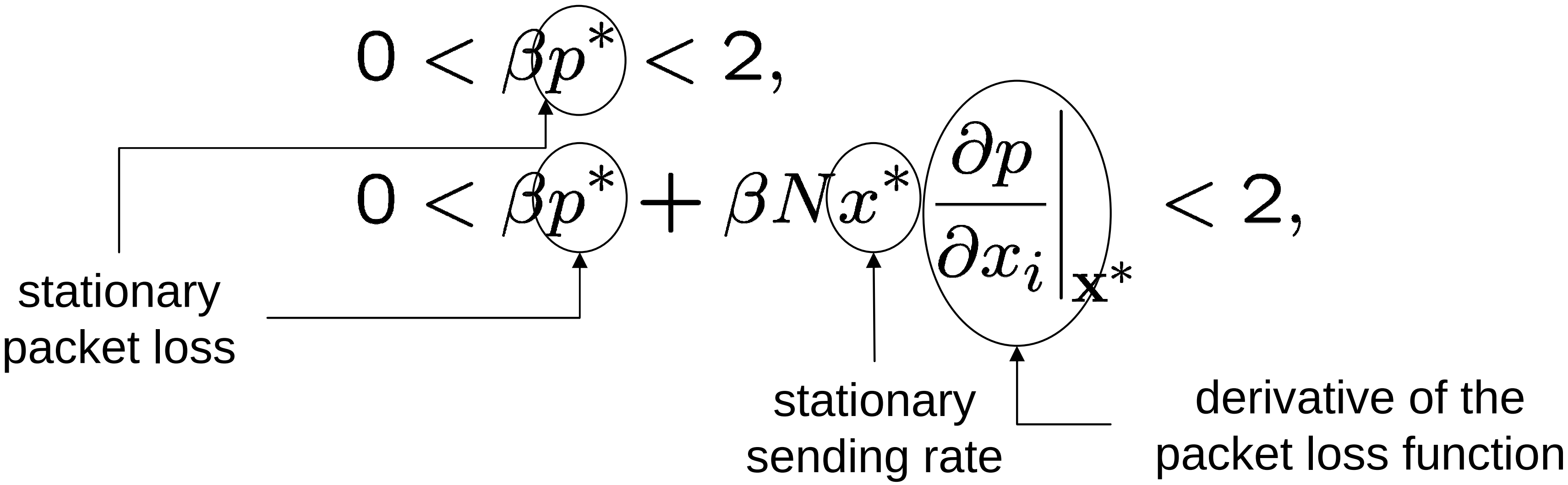
where $f_i(.)$ are some non-linear functions.
If the Jacobian matrix J is Hermitian, then system $\mathcal{N}_{\mathcal{D}}$ with arbitrary directional delays:

$$x_i(n) = f_i\Big(x_1(n - D_1^{\rightarrow} - D_i^{\leftarrow}), x_2(n - D_2^{\rightarrow} - D_i^{\leftarrow}), \\ \cdots, x_N(n - D_N^{\rightarrow} - D_i^{\leftarrow})\Big)$$

is stable if and only if $\mathcal{N}$ is stable

Stability of MKC

- The Jacobian of MKC is real and symmetric
- <u>Theorem</u>. Heterogeneously delayed MKC is locally asymptotically stable if and only if:



$$0 < \beta p^* < 2,$$

$$0 < \beta p^* + \beta N x^* \left.\frac{\partial p}{\partial x_i}\right|_{\mathbf{x}^*} < 2,$$

stationary
packet loss

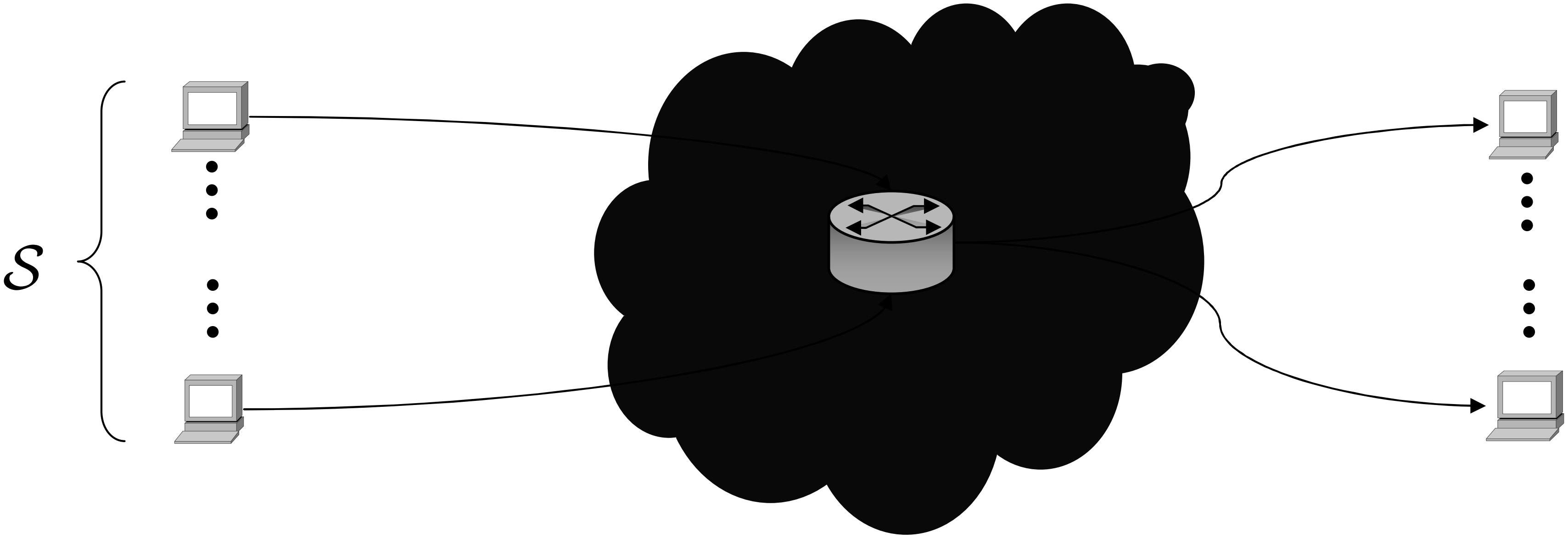
stationary
sending rate

derivative of the
packet loss function

Stability conditions do not depend on any delays or the routing matrix of end-flows!

Exponential MKC (EMKC)

- Assume a set $\mathcal{S}$ of N users congested by a common link of capacity C



- EMKC has a particular packet loss function $p(n)$:

$$p(n) = \frac{\sum_{u=1}^{N} x_u(n - D_u^{\rightarrow}) - C}{\sum_{u=1}^{N} x_u(n - D_u^{\rightarrow})}$$

Exponential MKC (EMKC) 2

- Theorem. Heterogeneously delayed EMKC is locally asymptotically stable if and only if $0 < \beta < 2$

The only parameter affecting heterogeneous stability of EMKC is β

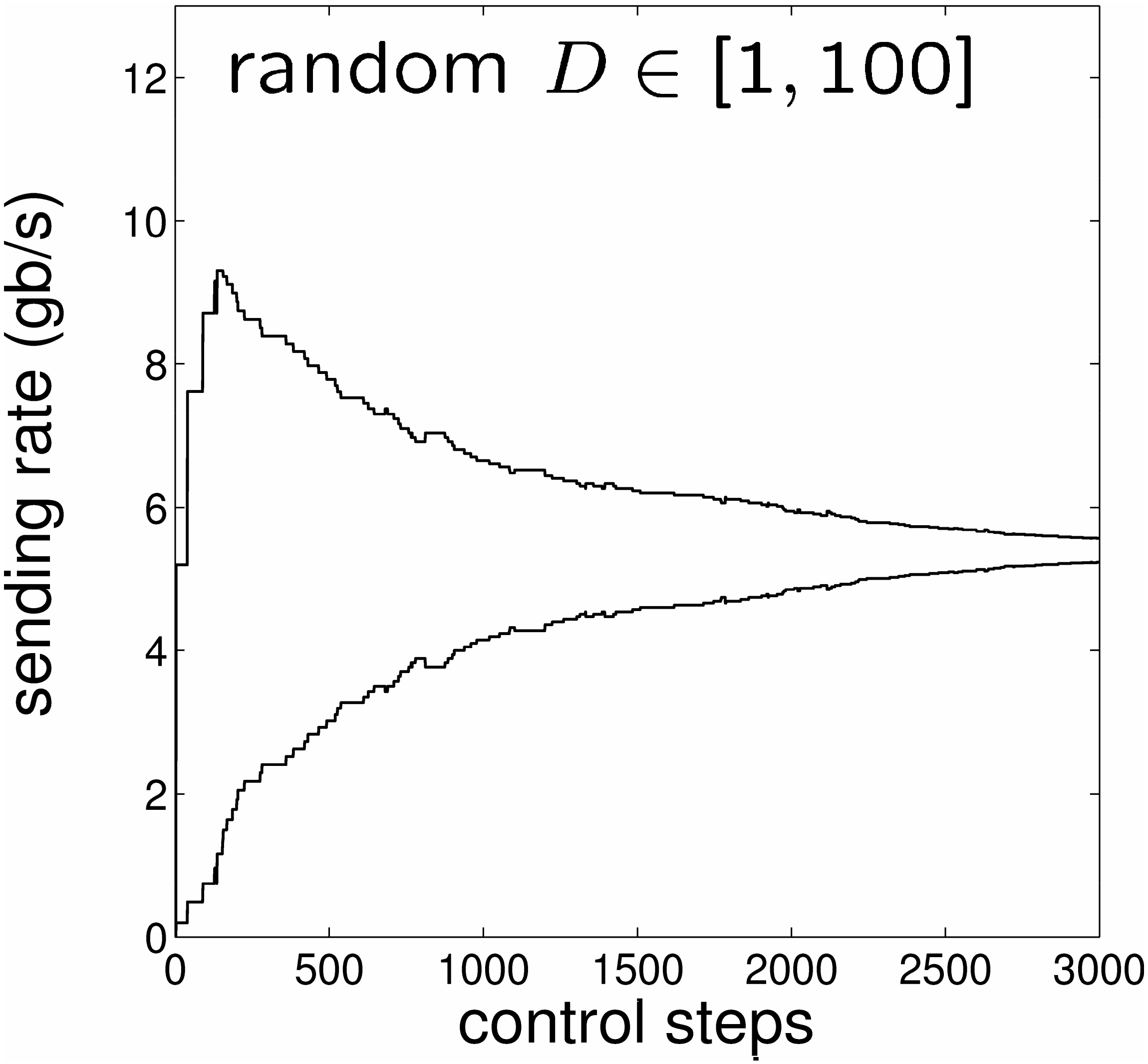
- In fact, many other systems with a symmetric Jacobian exhibit similar delay-independent stability
- The equilibrium individual rate is $x^* = C/N + \alpha/\beta$

EMKC is fair regardless of end-flow RTTs!

Exponential MKC (EMKC) 3

- Dynamics of EMKC under constant and random delays

For the same parameters, Kelly control is unstable for $D>3$



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Preliminaries

- Theorem. Assume a nonlinear system Ω:

$$x_n(n) = f(x_{n-1}, y_{n-1}),$$

where $f(.)$ is nonlinear, but in a special form:

$$f(x, y) = a + bx + cy + dxy.$$

Assume $y_n \to y^*$ as $n \to \infty$ and form system Ω' :

$$\tilde{x}_n = f(\tilde{x}_{n-1}, y^*).$$

Then system Ω converges if and only if system Ω' converges, in which case:

$$\lim_{n\to\infty} |x_n - \tilde{x}_n| = 0.$$

Global Stability of EMKC

- <u>Lemma</u>. When $0 < \beta < 2$, the combined rate $X(n)$ of EMKC is globally asymptotically stable under constant delay and converges to $X^* = C + N\alpha/\beta$ at an exponential rate

  Packet loss is expressed by:

  $$p(n) = \frac{X(n) - C}{X(n)}.$$

  Combining with the lemma, it is easy to obtain:

- <u>Corollary</u>. When $0 < \beta < 2$, the packet loss $p(n)$ of EMKC converges to $p^* = N\alpha/(C\beta + N\alpha)$ under constant delay regardless of initial conditions

Global Stability of EMKC 2

Combining the last corollary and the preliminary theorem, we have the following theorem

- Theorem. When $0 < \beta < 2$, individual flow rate $x_i(n)$ of an N-dimensional EMKC system converges to $x^* = C/N + \alpha/\beta$ under constant delay regardless of initial conditions
  - EMKC is globally quasi-asymptotically stable

Since EMKC is proven to be Lyapunov stable,

- Corollary. EMKC is globally asymptotically stable under homogeneous delay if and only if $0 < \beta < 2$.

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Conclusions

- There exists a set of nonlinear control systems whose local asymptotic stability is independent of feedback delay
- MKC exemplifies a class of controllers which are locally asymptotically stable regardless of delays and globally stable under constant delay
- Future work involves extension of these results to the multi-router case and Non-Hermitian Jacobian matrices